Dirac Operator on Noncommutative AdS_2

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Abstract

In this article we construct the chirality and Dirac operators on noncommutative AdS_2. We also derive the discrete spectrum of the Dirac operator which is important in the study of the spectral triple associated to AdS_2. It is shown that the degeneracy of the spectrum present in the commutative AdS_2 is lifted in the noncommutative case. The way we construct the chirality operator is suggestive of how to introduce the projector operators of the corresponding projective modules on this space.

1 Introduction

Recently there has been much interest in applying the Connes spectral triple approach to the study of noncommutative manifolds such as fuzzy sphere and noncommutative S^4 [1, 2]. In this approach one starts with an algebra $\mathcal{A}$ (say the algebra of functions on the underlying manifold X), a Hilbert space $\mathcal{H}$ which is a representation of $\mathcal{A}$, and a Dirac operator D which acts on $\mathcal{H}$. Having the triple $(\mathcal{A}, \mathcal{H}, D)$, the geometry and topology of X can be determined [3]. For example, the distance formula is given in terms of D, and knowing the spectrum of D, allows one to do the integration over X.

Fuzzy sphere is the simplest example of a curved noncommutative manifold for which the spectral triple has been explicitly worked out [4, 5]. Another two dimensional and non-flat example that comes to mind is the fuzzy (noncommutative) AdS_2. Fuzzy AdS_2 naturally appears in the study of a variant form of (noncommutative) AdS/CFT correspondence in

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two dimensions [6]. Note that the isometry group of AdS_2 is $SO(2,1)$, and to get the fuzzy version of AdS_2, one promotes the coordinates defining the embedding of the surface in R^{2+1} from ordinary functions to the generators of the group $SU(1,1)$. The embedding and the algebra satisfied by the coordinates are invariant under the automorphism group $SO(2,1)$ of the algebra and hence the name fuzzy AdS_2.

For fuzzy sphere, a knowledge of the Dirac operator and its spectrum has allowed the authors of [1] to compute the Chern numbers of the associated projective module $\mathcal{M}$ on fuzzy sphere. Here we will follow [4, 5] to construct the Dirac operator on fuzzy AdS_2, and derive its discrete spectrum. In particular, we will find that the discrete spectrum of D is nondegenerate. This should be contrasted with the commutative AdS_2 case where the discrete spectrum of D is degenerate. By this, we provide the spectral triple of fuzzy AdS_2. One may wish to build modules on this space, and, for instance, study the Yang-Mills theory on it. We introduce one such projective modules by constructing its corresponding projector on the fuzzy AdS_2. To compute the Chern number of these modules, however, one has to adapt the Connes formula to the case of noncompact algebras.

In the context of supersymmetric quantum mechanics on AdS_2, in [7] the spectrum of the Dirac operator, and the corresponding spinors for a charged particle with spin $\frac{1}{2}$ − in the presence of a magnetic monopole − was successfully derived. There it was shown that the Dirac spinors represent an $\mathcal{N}=1$ chiral supersymmetry algebra and a unitary parasupersymmetry algebra of an arbitrary order. Along the lines of [8, 9, 10], it is interesting to see how these results are modified on fuzzy AdS_2. This is one of our motivations to study the Dirac operator on fuzzy AdS_2.

2 Σ_3-pseudo-Hermiticity and AdS_2

In this section we study a special representation for the Lie algebra $su(1,1)$. This we choose to be a non-unitary representation given by 2×2 matrices. In this representation, the Σ_3-pseudo-Hermiticity − which will be defined shortly − structure of the algebra is very clear. Then we will introduce an appropriate inner product between the elements of the Hilbert space on which $su(1,1)$ acts. This will induce the Σ_3-pseudo-Hermiticity structure of the algebra on to the Hilbert space [11]. The Σ_3-pseudo-Hermiticity concept will become important when we come to discuss the spectrum of the Dirac operator on fuzzy AdS_2.

For every irreducible unitary finite dimensional representation of the compact semisimple Lie group $SU(2)$, one can analytically continue the representation to a finite dimensional necessarily non-unitary representation of the noncompact semisimple Lie group $SU(1,1)\cong SL(2,\mathbb{R})$. Both Lie groups $SU(2)$ and $SU(1,1)$ have a common maximal compact subgroup $U(1)$, and a common complexification which is the Lie group $SL(2,\mathbb{C})$. Considering the lowest dimensional irreducible representation of the Lie algebra $su(2)$, which is represented by the Pauli matrices σ_1, σ_2 and σ_3, one can construct a two dimensional non-unitary representation of the Lie algebra $su(1,1)$

$$\Sigma_1 = i\sigma_1 \qquad \Sigma_2 = i\sigma_2 \qquad \Sigma_3 = \sigma_3 \ . \tag{1}$$

It is straightforward to show that

$$\Sigma_i \Sigma_j = -\eta_{ij} I + i C_{ij}{}^k \Sigma_k \ , \tag{2}$$

where the indices are raised and lowered by $\eta_{ij} = (1, 1, -1)$. The structure constants C_{ijk}'s are completely anti-symmetric in their indices, and our convention is $C_{123} = 1$. If we parametrize the Lie group $SU(1,1)$ of 2×2 matrices of unit determinant as $U = exp(\frac{i}{2}\theta^i \Sigma_i)$, they will satisfy the following Σ_3-pseudo-unitary relation

$$U^\dagger \Sigma_3 U = U \Sigma_3 U^\dagger = \Sigma_3 \ . \tag{3}$$

In the next section, we will see how this property of Σ_3-pseudo-Hermiticity gets extended to the Dirac operator on noncommutative AdS_2. The relation (3) induces essentially the property of Σ_3-self-adjoint on 2×2 representation of the Lie algebra $su(1,1)$

$$\Sigma_i^\dagger = \Sigma_3 \Sigma_i \Sigma_3 \ \ , \tag{4}$$

where we note that the operator Σ_3 is self-adjoint, involutory and unitary

$$\Sigma_3^\dagger = \Sigma_3 \qquad \Sigma_3^2 = I \qquad \Sigma_3^{-1} = \Sigma_3^\dagger \ \ . \tag{5}$$

Therefore, the Lie algebra space $su(1,1)$ is said to be pseudo-Hermitian with respect to Σ_3 (or equivalently, we say $su(1,1)$ is Σ_3-pseudo-Hermitian). The eigenvalues of Σ_3-pseudo-Hermitian operators are known to be either real or appear in complex-conjugates pairs. Let the linear operator $u : \mathcal{H} \to \mathcal{H}$ be an arbitrary element of the Lie algebra $su(1,1)$ acting on a separable Hilbert space $\mathcal{H}$. u is expressed as a linear combination of the traceless 2×2 matrices Σ_1, Σ_2 and Σ_3 with real determinants. If we define the Σ_3-adjoint for every Ψ belonging to the Hilbert space $\mathcal{H}$ as

$$\overline{\Psi} = \Psi^\dagger \Sigma_3 \ , \tag{6}$$

then we can introduce a natural indefinite inner product "$*$" of the two arbitrary elements Ψ and Φ belonging to the Hilbert space $\mathcal{H}$, as

$$\Psi * \Phi = \overline{\Psi}\Phi \ . \tag{7}$$

This has the following property for every arbitrary element u of the Lie algebra $su(1,1)$

$$\Psi * (u\Phi) = \overline{\Psi} u \Phi = \Psi^\dagger \Sigma_3 u \Phi = \Psi^\dagger u^\dagger \Sigma_3 \Phi = (u\Psi)^\dagger \, \Sigma_3 \Phi = \overline{(u\Psi)}\Phi = (u\Psi) * \Phi \ , \tag{8}$$

which means that all the generators belonging to $su(1,1)$ are Σ_3-pseudo-Hermitian with respect to the inner product "$*$". A Hilbert space equipped with such an indefinite inner product is called the Krein space [12]. Using Eq. (2) the Lie and Clifford algebras corresponding to the generators Σ_i are obtained, respectively as

$$[\Sigma_i, \Sigma_j] = 2i C_{ij}{}^k \Sigma_k \tag{9}$$

$$\{\Sigma_i, \Sigma_j\} = -2\eta_{ij} I \ . \tag{10}$$

It is straightforward to show that the structure constants ${C_{ij}}^k$ satisfy the following relation

$$ {C_{im}}^k \eta^{ij} {C_{jl}}^n = {\eta_m}^n {\eta_l}^k - \eta_{ml}\eta^{kn}. \tag{11} $$

Next let us discuss the geometrical structure of the Euclidean AdS_2. The Euclidean $AdS_2 = SU(1,1)/U(1)$ is defined through the following embedding in 3-dimensional flat Minkowskian space ($l > 0$)

$$ \mathbf{x} \cdot \mathbf{x} := x_i \eta^{ij} x_j = -l^2 \ . \tag{12} $$

This could also be thought of as the poincare upper half-plane $H^2 = \{(x, y) \in I\!R^2 : y > 0\}$ with the Riemannian metric

$$ ds^2 = l^2 \frac{dx^2 + dy^2}{y^2} \ . \tag{13} $$

If we make the following coordinates transformations

$$ x = \frac{2\tanh\frac{\rho}{2}\sin\tau}{1 + 2\tanh\frac{\rho}{2}\cos\tau + \tanh^2\frac{\rho}{2}} \qquad\qquad y = \frac{1 - \tanh^2\frac{\rho}{2}}{1 + 2\tanh\frac{\rho}{2}\cos\tau + \tanh^2\frac{\rho}{2}} \ , \tag{14} $$

the metric (13) will transform to

$$ ds^2 = l^2(d\rho^2 + \sinh^2\rho \, d\tau^2) \ . \tag{15} $$

The above metric can also be obtained by simply inserting the embedding coordinates

$$ \begin{aligned} x_1 &= l \sinh\rho\cos\tau \\ x_2 &= l \sinh\rho\sin\tau \\ x_3 &= l \cosh\rho \end{aligned} \tag{16} $$

into the Minkowskian metric

$$ ds^2 = dx_1^2 + dx_2^2 - dx_3^2 \, . \tag{17} $$

Also note that the metric (15) has a negative scalar curvature $R = -2/l^2$.

3 $U(1)$ principal fiber bundles over AdS_2

For the sake of completeness, here we discuss the Hopf construction of the $U(1)$ bundles over AdS_2. This will prove useful in the discussion of the noncommutative analogue of bundles over AdS_2, i.e., the construction of projective modules over fuzzy AdS_2.

The $U(1)$ principal fiberation π of the total space AdS_3 over AdS_2 can be realized as follows. First we define the total space in $\mathbb{C}^2$ through the embedding

$$ AdS_3 = \left\{ (z_1, z_2) \in \mathbb{C}^2, \quad |z_1|^2 - |z_2|^2 = -l^2 \right\} \ . \tag{18} $$

In the framework of principal fiber bundle, this is projected over the base manifold AdS_2 as

$$U(1) \overset{\text{right } U(1)\text{–action}}{\hookrightarrow} AdS_3 \xrightarrow{\pi} AdS_2 \,. \tag{19}$$

The right $U(1)$-action transforms the point (z_1, z_2) of AdS_3 onto another point of AdS_3

$$\left.\begin{array}{l} AdS_3 \times U(1) \longrightarrow AdS_3 \\ (z_1, z_2)\, w = (z_1 w, z_2 w) \end{array}\right\} \rightarrow |z_1 w|^2 - |z_2 w|^2 = |z_1|^2 - |z_2|^2 = -l^2 \,. \tag{20}$$

The complex Hopf bundle projection $\pi : AdS_3 \longrightarrow AdS_2$ is given by

$$\begin{aligned} x_1 &= \frac{1}{l}\left(z_1 \bar{z}_2 + z_2 \bar{z}_1\right) \\ x_2 &= \frac{i}{l}\left(z_1 \bar{z}_2 - z_2 \bar{z}_1\right) \\ x_3 &= \frac{1}{l}\left(|z_1|^2 + |z_2|^2\right) \,. \end{aligned} \tag{21}$$

The relation (12) can be directly checked. The sections of this $U(1)$ bundle over the base manifold AdS_2 are then the inverse of (21), i.e.,

$$|z_1|^2 = \frac{l}{2}\left(x_3 - l\right) \qquad |z_2|^2 = \frac{l}{2}\left(x_3 + l\right) \qquad z_1 \bar{z}_2 = \frac{l}{2}\left(x_1 - i x_2\right) \,. \tag{22}$$

Moreover, the Hopf fibration of $U(1)$ bundles over AdS_2 can also be expressed in the group theoretical framework as follows. There is a canonical one-to-one correspondence between the points on AdS_3 and the elements of $SU(1,1)$, $g : AdS_3 \longrightarrow SU(1,1)$. It is sufficient to choose the one-to-one map g as

$$g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{l} \begin{pmatrix} \bar{z}_2 & i z_1 \\ -i \bar{z}_1 & z_2 \end{pmatrix} =: U \,, \tag{23}$$

in which z_1 and z_2 are arbitrary elements of AdS_3. It is easy to check that it has the unit determinant, and also, satisfies Σ_3-pseudo-unitary relation (3). The homogeneous manifold $SU(1,1)/U(1)$ can be obtained by the Hopf projection as

$$\pi\left(U\right) := l U \Sigma_3 U^{-1} = -\mathbf{x} \cdot \mathbf{\Sigma} \,, \tag{24}$$

where the coordinates x_i (for homogeneous manifold $SU(1,1)/U(1)$) are obtained from the complex coordinates z_1 and z_2 (for the group manifold $SU(1,1)$) via Eqs. (21). The right $U(1)$-action over $SU(1,1)$ keeps the base point $\mathbf{x}$ fixed in the sence that all the elements $U exp\left(\frac{i}{2}\theta^3 \Sigma_3\right)$ of $SU(1,1)$ are projected onto the same point $\mathbf{x}$ on the base manifold $SU(1,1)/U(1)$. Therefore, we can identify any equivalance class $[U] = U exp\left(\frac{i}{2}\theta^3 \Sigma_3\right) \in SU(1,1)/U(1)$ with the point $\mathbf{x} \in AdS_2$ via the projection map (24).

4 Noncommutative AdS_2 and finite dimensional Schwinger representation of the algebra $\mathcal{A}_N$

As mentioned in introduction, to define the fuzzy AdS_2, we promote the coordinates x_i's of AdS_2 to play the role of the generators of $su(1,1)$, in some unitary representation. For a given integer N ($N > 2$), let ($\mathcal{A}_N$,$\mathcal{L}_N$) denote the space of all analytic functions of coordinates, and derivations on fuzzy AdS_2, respectively. As in the case of fuzzy sphere we can work out a Schwinger representation for $\mathcal{A}_N$. To start with, introduce a real parameter α controling the strength of the noncommutativity, and take the coordinates x_i's to satisfy the relation

$$[x_i, x_j] = i\alpha C_{ij}{}^k x_k \,, \tag{25}$$

such that $\frac{2}{\alpha}x^i$'s satisfy the commutation relations of the Lie algebra $su(1,1)$. Like $SU(2)$, the Lie group $SU(1,1)$ is a group of rank 1, and so possesses one invariant Casimir operator whose eigenvalues label the irreducible representations. The noncommutative relation (25) together with the embedding (12), which now ascribes a negative constant value to the Casimir operator, defines the noncommutative AdS_2. In fact, this is an analytic continuation of the noncommutative structure of the fuzzy sphere to that of AdS_2 via the metric η_{ij}. The passage from the fuzzy sphere to the noncommutative AdS_2 is equivalent analytically to the passage from the ordinary Euclidean Clifford algebra to the Minkowskian Clifford algebra.

The generators L_i of $\mathcal{L}_N$ are defined by the adjoint action of x_i on the space $\mathcal{A}_N$:

$$\frac{1}{\alpha} ad_{x_i} x_j = \frac{1}{\alpha}[x_i, x_j] =: L_i x_j \,. \tag{26}$$

This induces the commutation relations of the Lie algebra $su(1,1)$ on $\mathcal{L}_N$

$$[L_i, L_j] = iC_{ij}{}^k L_k \,. \tag{27}$$

The bigger algebra ($\mathcal{A}_N$, $\mathcal{L}_N$) includes the commutation relations (25) and (27), as well as

$$[L_i, x_j] = iC_{ij}{}^k x_k \,. \tag{28}$$

We are now going to introduce a version of the Schwinger operator bases which realizes a representation of the Lie algebra $su(1,1)$ using the Σ_3-pseudo-Hermitian matrices (1). In this formulation, one considers the elements of $\mathcal{A}_N$ acting on a finite dimensional Hilbert space $\mathcal{F}_N$ spanned by a set of orthonormal bases $\{|k\rangle\}_{k=0}^{k=N-2}$. Then, the algebra $\mathcal{A}_N$ can be identified with the algebra of $(N-1)\times(N-1)$ complex matrices, which act on an $(N-1)$-dimensional Hilbert space $\mathcal{F}_N$. According to the discussion above, the Hilbert space $\mathcal{F}_N$ can be constructed by acting a pair of creation and annihilation operators $\mathbf{a}^\dagger{}_b$ and $\mathbf{a}^b$ ($b = 1, 2$) on the vacuum state $|0\rangle$, i.e.,

$$|k\rangle = \frac{1}{\sqrt{k!(N-2-k)!}} \left(\mathbf{a}^\dagger{}_1\right)^k \left(\mathbf{a}^\dagger{}_2\right)^{N-2-k} |0\rangle \qquad\qquad k = 0, 1, \cdots, N-2 \,, \tag{29}$$

where the commutation relations are

$$\left[\mathbf{a}^a, \mathbf{a}^b\right] = \left[\mathbf{a}^\dagger{}_a, \mathbf{a}^\dagger{}_b\right] = 0 \qquad\qquad \left[\mathbf{a}^a, \mathbf{a}^\dagger{}_b\right] = \delta^a_b\,. \tag{30}$$

The number operator $\mathbf{N} := \mathbf{a}^\dagger{}_b \mathbf{a}^b$ has the eigenvalue $N-2$ over the Hilbert space $\mathcal{F}_N$

$$\mathbf{N}|_{\mathcal{F}_N} = N - 2\,. \tag{31}$$

The Schwinger representation for the bases of the algebra $\mathcal{A}_N$ is now represented as follows

$$x_i = \frac{1}{2}\alpha\,(\Sigma_i)^a{}_b\,\mathbf{a}^\dagger{}_a \mathbf{a}^b\,. \tag{32}$$

It is straightforward to see that the generators (32) satisfy the algebra of noncommutative AdS_2 (25), as well as the following commutation relations:

$$\begin{aligned}\left[x_i, \mathbf{a}^\dagger{}_a\right] &= \frac{1}{2}\alpha\,(\Sigma_i)^b{}_a\,\mathbf{a}^\dagger{}_b \\ \left[x_i, \mathbf{a}^a\right] &= \frac{-1}{2}\alpha\,(\Sigma_i)^a{}_b\,\mathbf{a}^b\,.\end{aligned} \tag{33}$$

The Casimir operator of the generators (32) is

$$\mathbf{x}\cdot\mathbf{x} = \frac{-\alpha^2}{4}\mathbf{N}(\mathbf{N}+2)\,. \tag{34}$$

Restricting the equations over the Hilbert space $\mathcal{F}_N$, and comparing Eq. (12) with (34), we conclude that

$$\alpha = \frac{2l}{\sqrt{N(N-2)}}\,, \tag{35}$$

which implies that the commutative limit $(\alpha \to 0)$ corresponds to the limit $N \to \infty$.

5 Chirality and Dirac operators

Our aim in this section is to construct the Dirac operator on fuzzy AdS_2. This is the most important ingredient in the Connes construction of noncommutative manifolds. To define the Dirac operator, however, first we need to introduce the chirality operator γ. The existence of this latter operator provides a Z_2 grading of the Hilbert space $\mathcal{H}$. γ has the following properties:

- Commutes with the elements of the algebra $\mathcal{A}_N$, and squares to one.
- Has a standard commutative limit.

This chirality operator, however, instead of being Hermitian as in the case of compact algebras, turns out to be Σ_3-pseudo-Hermitian on fuzzy AdS_2. The Dirac operator D is then constructed such that:

• It anticommutes with γ.

• Reduces to the conventional Dirac operator on commutative AdS_2 [10], when the non-commutativity parameter α is sent to zero.

Notice that the above requirements on γ and D do not uniquely fix their representations. Let us represent the Dirac and chirality operators by the 2-component spinors $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ as the elements of $\mathcal{A}_N$-bimodule $\mathcal{H} := \mathbb{C}^2 \otimes \mathcal{A}_N$.

Following [5], to construct γ, we introduce the opposite algebra $\mathcal{A}_N^0$ with the product rule between the elements

$$x_i^0 x_j^0 \equiv (x_j x_i)^0 \,. \tag{36}$$

In other words, the generators x_i^0, which form the opposite algebra $\mathcal{A}_N^0$, are acting from the right hand side on $\mathcal{A}_N$-bimodule:

$$x_i^0 \Psi = \Psi x_i \qquad\qquad \forall \Psi \in \mathcal{H}. \tag{37}$$

From these, we obtain the commutation relation of this algebra to be

$$\left[x_i^0, x_j^0\right] = -i\alpha C_{ij}{}^k x_k^0 \,. \tag{38}$$

Now let us define the chirality operator to be

$$\gamma = \frac{1}{\mathcal{N}_N}(\Sigma_i \eta^{ij} x_j^0 + \frac{\alpha}{2} I) \,, \tag{39}$$

where

$$\mathcal{N}_N = \sqrt{l^2 + \frac{\alpha^2}{4}} = \frac{l(N-1)}{\sqrt{N(N-2)}} \,. \tag{40}$$

It is easy to check that γ is involutory, just note that

$$\begin{aligned} (\Sigma_i \eta^{ij} x_j^0)(\Sigma_k \eta^{kl} x_l^0) &= \eta^{ij}\eta^{kl}(-\eta_{ik} I + i C_{ik}{}^n \Sigma_n) x_j^0 x_l^0 \\ &= -\eta^{jl} x_j^0 x_l^0 I + i C^{jln} \Sigma_n x_j^0 x_l^0 \\ &= l^2 I + \frac{\alpha}{2} C^{jln} C_{jl}{}^r \Sigma_n x_r^0 \\ &= l^2 I - \alpha \eta^{nr} \Sigma_n x_r^0 \,, \end{aligned} \tag{41}$$

so

$$\gamma^2 = \frac{1}{\mathcal{N}_N^2}\left(\frac{\alpha^2}{4} I + \alpha \Sigma_i \eta^{ij} x_j^0 + l^2 I - \alpha \Sigma_i \eta^{ij} x_j^0\right) = I \,. \tag{42}$$

Note that the chirality operator γ is constructed so that it commutes with the elements of algebra $\mathcal{A}_N$. Moreover, since $\Sigma_i^\dagger = \Sigma_3 \Sigma_i \Sigma_3$, it is Σ_3-pseudo-Hermitian

$$\gamma^\dagger = \Sigma_3 \gamma \Sigma_3 \,. \tag{43}$$

Next we introduce the Dirac operator

$$D = \frac{-i}{l\alpha}\gamma\, C_{ij}{}^{k}\eta^{il}\eta^{jm}\Sigma_l x_m^0 x_k\,. \tag{44}$$

In the following we show that γ anticommutes with D. First note that

$$\begin{aligned}(\eta^{ij}\Sigma_i x_j^0)(\Sigma_l x_k^0) = \eta^{ij}\Sigma_i\Sigma_l x_j^0 x_k^0 &= \eta^{ij}(-\Sigma_l\Sigma_i - 2\eta_{il}I)(x_k^0 x_j^0 - i\alpha C_{jk}{}^{r} x_r^0) \\ &= -(\Sigma_l x_k^0)(\eta^{ij}\Sigma_i x_j^0) + i\alpha\eta^{ij}C_{jk}{}^{r}\Sigma_l\Sigma_i x_r^0 - 2x_l^0 x_k^0 I\,, \end{aligned} \tag{45}$$

where in the first line we have used Eq. (10), and the relation (38). So we learn that

$$\begin{aligned}\left\{C_{ij}{}^{k}\eta^{il}\eta^{jm}\Sigma_l x_m^0\,, \eta^{ij}\Sigma_i x_j^0\right\} &= i\alpha C_{ij}{}^{k}C_{sm}{}^{p}\eta^{il}\eta^{jm}\eta^{rs}\Sigma_l\Sigma_r x_p^0 - 2C_{ij}{}^{k}\eta^{il}\eta^{jm}x_l^0 x_m^0 I \\ &= -\alpha C_{ij}{}^{k}\eta^{il}C_{lr}{}^{n}\eta^{jm}\eta^{rs}C_{sm}{}^{p}\Sigma_n x_p^0 \\ &= -\alpha\eta^{nm}\eta^{ks}C_{sm}{}^{p}\Sigma_n x_p^0 \\ &= -\alpha\eta^{ni}\eta^{pj}C_{ij}{}^{k}\Sigma_n x_p^0\,. \end{aligned} \tag{46}$$

The second and the third equality follow from Eqs. (2) and (11). Since x_i and x_j^0 commute with each other, we conclude that

$$\begin{aligned}\{D\,,\gamma\} &= \frac{-i}{l\alpha\mathcal{N}_N}\gamma\left\{C_{ij}{}^{k}\eta^{il}\eta^{jm}\Sigma_l x_m^0 x_k\,, \eta^{ij}\Sigma_i x_j^0\right\} + \frac{\alpha}{\mathcal{N}_N}D \\ &= \frac{i}{l\mathcal{N}_N}\gamma C_{ij}{}^{k}\eta^{il}\eta^{jm}\Sigma_l x_m^0 x_k + \frac{\alpha}{\mathcal{N}_N}D \\ &= 0\,. \end{aligned} \tag{47}$$

Further, Eq. (43) implies that

$$D^\dagger = \Sigma_3 D\Sigma_3\,, \tag{48}$$

so it is Σ_3-pseudo-Hermitian.

To proceed further and to discuss the commutative limit of the Dirac operator, there are some definitions and algebra among the operators in $(\mathcal{A}_N, \mathcal{L}_N)$ that should be discussed. The commutation relations (27), (38) and also

$$\left[L_i, x_j^0\right] = iC_{ij}{}^{k}x_k^0 \tag{49}$$

which is obtained from Eqs. (26) and (37), define the algebra $(\mathcal{A}_N^0, \mathcal{L}_N)$ for the noncommutative AdS_2. Note that the generators of the algebra $\mathcal{L}_N$ can also be written as a linear combination of the generators of the algebras $\mathcal{A}_N$ and $\mathcal{A}_N^0$ as

$$L_i = \alpha^{-1}\left(x_i - x_i^0\right). \tag{50}$$

Introduce the following operators in $(\mathcal{A}_N, \mathcal{L}_N) \otimes M_2(\mathbb{C})$

$$\chi := x_i\eta^{ij}\Sigma_j \tag{51}$$
$$\Lambda := L_i\eta^{ij}\Sigma_j \tag{52}$$
$$\Sigma := -iC_{ij}{}^{k}\eta^{il}\eta^{jm}x_l L_m\Sigma_k\,, \tag{53}$$

which are invariant under the $SU(1,1)$ transformations.[1] Using Eq. (4), one can easily show that

$$\begin{aligned}
\chi^\dagger &= \Sigma_3 \chi \Sigma_3 && (54)\\
\Lambda^\dagger &= \Sigma_3 \Lambda \Sigma_3 && (55)\\
\Sigma^\dagger &= -\Sigma_3 \left(\Sigma + 2\chi\right) \Sigma_3 \,. && (56)
\end{aligned}$$

If we define[2]

$$J_i = L_i + \frac{1}{2}\Sigma_i \,, \tag{57}$$

and

$$\Omega_i = C_{ij}{}^k \eta^{jl} x_l \Sigma_k \tag{58}$$

then, using Eqs. (2), (11) and (25), it follows that

$$C_{ij}{}^k \eta^{il} \eta^{jm} \Sigma_l x^0_m x_k = -\alpha \mathbf{\Omega} \cdot \mathbf{J} = i\alpha \left(\Sigma + \chi\right) \,. \tag{59}$$

[1] Using the commutation relations of the algebra $(\mathcal{A}_N, \mathcal{L}_N)$, the (anti)commutation relations of the operators χ, Λ and Σ are

$$\begin{aligned}
\frac{1}{2}\{\chi,\chi\} &= l^2 + \alpha\chi\\
\frac{1}{2}\{\Lambda,\Lambda\} &= -\mathbf{L}\cdot\mathbf{L} + \Lambda\\
\frac{1}{2}\{\Sigma,\Sigma\} &= -\alpha\mathbf{x}\cdot\mathbf{L} + l^2\mathbf{L}\cdot\mathbf{L} + (\mathbf{x}\cdot\mathbf{L})^2 - \alpha\Sigma + \alpha\mathbf{x}\cdot\mathbf{L}\Lambda + l^2\Lambda \,,
\end{aligned}$$

$$\begin{aligned}
\{\chi,\Lambda\} &= 2\left(\chi - \mathbf{x}\cdot\mathbf{L}\right)\\
[\chi,\Lambda] &= -2\left(\chi + \Sigma\right) \,,
\end{aligned}$$

$$\begin{aligned}
\{\Sigma,\Lambda\} &= 2\left(\Sigma + \mathbf{x}\cdot\mathbf{L}\right)\\
[\Sigma,\Lambda] &= 2\left(\chi\mathbf{L}\cdot\mathbf{L} - \Lambda\mathbf{x}\cdot\mathbf{L}\right) \,,
\end{aligned}$$

$$\begin{aligned}
\{\Sigma,\chi\} &= -2l^2 + \alpha\left(\Sigma - \chi\right)\\
[\Sigma,\chi] &= -2l^2\left(I - \Lambda\right) - 2\alpha\mathbf{x}\cdot\mathbf{L} + \{\chi, \mathbf{x}\cdot\mathbf{L}\} \,.
\end{aligned}$$

[2] Note that the generators J_i are Σ_3-pseudo-Hermitian with respect to the inner product "$*$" defined in the $\mathcal{A}_N$-bimodule $\mathcal{H} := \mathbb{C}^2 \otimes \mathcal{A}_N$, because

$$\Psi * (J_i\Phi) = \Psi^\dagger \Sigma_3 \left(L_i + \frac{1}{2}\Sigma_i\right)\Phi = \Psi^\dagger \left(L_i + \frac{1}{2}\Sigma_i^\dagger\right)\Sigma_3\Phi = \left(\left(L_i + \frac{1}{2}\Sigma_i\right)\Psi\right)^\dagger \Sigma_3\Phi = (J_i\Psi) * \Phi \,.$$

Also note that the generators J_i, which like the bases of $\mathcal{A}_N$ and $\mathcal{L}_N$, constitute the Lie algebra $su(1,1)$, satisfy the following commutation relations

$$[L_i, J_j] = iC_{ij}{}^k L_k \,.$$

Therefore the Dirac operator can be written in the following compact forms

$$D = \frac{i}{l}\gamma \mathbf{\Omega} \cdot \mathbf{J} = \frac{1}{l}\gamma\left(\Sigma + \chi\right) . \tag{60}$$

In this form the Σ_3-pseudo-Hermiticity relation (48) can easily be derived from the second equality in (60) upon taking its dagger and using Eqs. (42), (43), (47), (54) and (56).

Now let us discuss the status of the Casimir operators in $\mathcal{L}_N$. Unlike the commutative case where the only Casimir is $\mathbf{L} \cdot \mathbf{L}$, here we get an additional Casimir operator. Though, it will scale to zero in the commutative limit $\alpha \to 0$. First of all, it is clear that the operator $\mathbf{L} \cdot \mathbf{L}$, the Casimir operator of the bases of the algebra $\mathcal{L}_N$, satisfies the following relation

$$[L_i, \mathbf{L} \cdot \mathbf{L}] = 0 \,. \tag{61}$$

On the other hand, from the commutation relations (27) and (28), one obtains

$$[L_i, \mathbf{x} \cdot \mathbf{L}] = 0 \,, \tag{62}$$

giving the second Casimir mentioned above. If we define the generalized momentum operators

$$P_i = -iC_{ij}{}^{k}\eta^{jl}x_l L_k \,, \tag{63}$$

which are invariant under the $SU(1,1)$ group transformations, we can obtain the following relations in the algebra $(\mathcal{A}_N, \mathcal{L}_N)$

$$[x_i, \mathbf{L} \cdot \mathbf{L}] = 2\left(x_i + P_i\right) \tag{64}$$

$$[x_i, \mathbf{x} \cdot \mathbf{L}] = \alpha\left(x_i + P_i\right) . \tag{65}$$

And finally, one can obtain the action of both operators $\mathbf{L} \cdot \mathbf{L}$ and $\mathbf{x} \cdot \mathbf{L}$ on the polynomials in the algebra $\mathcal{A}_N$ using the inductive procedure ($n = 1, 2, 3, \cdots$)

$$\mathbf{L} \cdot \mathbf{L}\left(x_i\right)^n = -2n\left(x_i\right)^n \tag{66}$$

$$\mathbf{x} \cdot \mathbf{L}\left(x_i\right)^n = -n\alpha\left(x_i\right)^n . \tag{67}$$

Comparing Eqs. (61) and (62), (64) and (65), and also, (66) and (67), it is readily found that

$$\mathbf{x} \cdot \mathbf{L} = \frac{\alpha}{2}\mathbf{L} \cdot \mathbf{L} \,, \tag{68}$$

which shows that the commutative limit is reached when $\alpha \to 0$.

We notice that in the commutative limit $\alpha \to 0$ (or $N \to \infty$), the generators of the opposite algebra, x_i^0's, get transformed into the generators of $\mathcal{A}_N$, i.e. x_i's. Therefore, using Eqs. (2) and (11) we get

$$D_\infty = -\left(\Lambda - I\right) - \frac{1}{l^2}\chi \mathbf{x} \cdot \mathbf{L} \,. \tag{69}$$

However, taking into account the relation (68), the second term of D_∞ also vanishes, and we are left with

$$D_\infty = -\left(\Lambda - I\right) , \tag{70}$$

which is in agreement with the results of [10]. Therefore Dirac operator (44) reduces to the conventional Dirac operator on commutative AdS_2 when the noncommutativity parameter α is sent to zero.

Before concluding this section, let us briefly discuss the related subject of projective modules on fuzzy AdS_2. Projective modules are analogue of fiber bundles on noncommutative spaces. So if we are to study, for instance, the Yang-Mills theory on noncommutative spaces, the concept of projective modules becomes indispensable. To construct a projective module, of a rank say 2, we proceed as follows. First we introduce a projector p, which is a 2×2 matrix with elements in the algebra $\mathcal{A}$. This, in turn, allows us to define the projective module $\mathcal{M} = p\,\mathcal{A}^2$ of sections $s \in \mathcal{A}^2$ such that $p \circ s = s$. The connection $\nabla = p \circ d$ on $\mathcal{M}$ is then a map from sections to one-form valued sections in $\mathcal{M}$.

From what we did in the case of chirality operator γ, which squares to I, it is not difficult to guess the form of p. We define

$$p = \frac{1}{2}(I + \Pi)\,, \tag{71}$$

where

$$\Pi = \frac{1}{\mathcal{N}_N}(\widetilde{\Sigma}_i \eta^{ij} x_j - \frac{\alpha}{2} I)\,, \tag{72}$$

such that $\Pi^2 = I$, therefore $p^2 = p$ and p is a projector. Note that in the above, $\widetilde{\Sigma}_i$'s are again the 2-dimensional representations of the $su(1,1)$ generators such that $[\widetilde{\Sigma}_i, \Sigma_j] = 0$. Now, as mentioned earlier, p defines the projective module $\mathcal{M} = p\,\mathcal{A}^2$ over $\mathcal{A}$ with the sections $s \in \mathcal{A}^2$ such that $p \circ s = s$. Let us introduce the differential operator d through

$$du = i[D, u]\,, \tag{73}$$

for any u in the Lie algeba of $su(1,1)$. If s is a section of the projective module $\mathcal{M}$ then $s = p \circ s$, and we define the connection on $\mathcal{M}$ to be $\nabla = d \circ p$. Therefore the curvature of $\mathcal{M}$ is $\nabla^2 = pdp^2$ with the first Chern class of

$$C_1(p) = \mathrm{tr}\,(pdp^2)\,. \tag{74}$$

For compact algebras, the first Chern number of $\mathcal{M}$ is then calculated by taking the Dixmier trace over the Hilbert space $\mathcal{H}$. The formula for the first Chern number is [3]

$$c_1(p) = -\mathrm{Tr}_w \left(\frac{1}{|D|^2} \gamma\, p[D, p][D, p] \right)\,, \tag{75}$$

where $|D| = \sqrt{D^\dagger D}$. For the projective module defined by the projector (71), it is straightforward to compute the first Chern class defined in (74). But to calculate the first Chern number and to see whether the associated module is nontrivial needs a modification of the Connes formula (75) to the case of noncompact algebras. This is a problem that we would like to further study in future works.

6 The discrete spectrum of Dirac operator on noncommutative AdS_2

In order to analyze the spectrum of the Dirac operator on noncommutative AdS_2, first note that using Eqs. (12) and (35) we can write

$$\mathbf{X}\cdot\mathbf{X}=\mathbf{X^0}\cdot\mathbf{X^0}=\frac{-N}{2}\left(\frac{N}{2}-1\right), \tag{76}$$

where

$$X_i=\alpha^{-1}x_i \qquad\qquad X_i^0=-\alpha^{-1}x_i^0 . \tag{77}$$

The square of the Dirac operator, on the other hand, is calculated by using Eqs. (2), (11), (25), (38), and the definition (44)

$$\frac{l^2}{\alpha^2}D^2=\mathbf{X}\cdot\mathbf{X}\,\mathbf{X^0}\cdot\mathbf{X^0}-\mathbf{X}\cdot\mathbf{X^0}\left(\mathbf{X}\cdot\mathbf{X^0}-I+\mathbf{\Sigma}\cdot\left(\mathbf{X}+\mathbf{X^0}\right)\right). \tag{78}$$

This can be simplified further using (50) and (57)

$$\mathbf{X}\cdot\mathbf{X^0}=\frac{1}{2}\left(\mathbf{L}\cdot\mathbf{L}-\mathbf{X}\cdot\mathbf{X}-\mathbf{X^0}\cdot\mathbf{X^0}\right) \tag{79}$$

$$\mathbf{\Sigma}\cdot\left(\mathbf{X}+\mathbf{X^0}\right)=\mathbf{J}\cdot\mathbf{J}-\mathbf{L}\cdot\mathbf{L}+\frac{3}{4}I, \tag{80}$$

the last equation is obtained noticing $\left(\frac{1}{2}\mathbf{\Sigma}\right)\cdot\left(\frac{1}{2}\mathbf{\Sigma}\right)=-s(s-1)I$ with $s=\frac{-1}{2}$. Finally, applying Eqs. (79) and (80), the square of Dirac operator transforms to the form

$$\begin{aligned}\frac{l^2}{\alpha^2}D^2 &= \mathbf{X}\cdot\mathbf{X}\,\mathbf{X^0}\cdot\mathbf{X^0}-\frac{1}{2}\left(\mathbf{L}\cdot\mathbf{L}-\mathbf{X}\cdot\mathbf{X}-\mathbf{X^0}\cdot\mathbf{X^0}\right)\\ &\quad\times\left[\frac{1}{2}\left(\mathbf{L}\cdot\mathbf{L}-\mathbf{X}\cdot\mathbf{X}-\mathbf{X^0}\cdot\mathbf{X^0}\right)+\mathbf{J}\cdot\mathbf{J}-\mathbf{L}\cdot\mathbf{L}-\frac{1}{4}I\right].\end{aligned} \tag{81}$$

There are four classes of irreducible representations of the Lie algebra $su(1,1)$. The first and the second classes include two principal discrete representations, which are realized in the Hilbert space through

$$\mathcal{D}^{\pm}(j)=\{|j,m_j\rangle:\quad j>0,\quad m_j=\pm j,\pm(j+1),\pm(j+2),\cdots\} \tag{82}$$

where

$$\mathbf{J}^2\,|j,m_j\rangle=-j(j-1)\,|j,m_j\rangle, \qquad\qquad J_3\,|j,m_j\rangle=m_j\,|j,m_j\rangle . \tag{83}$$

It is clear that the state $|j,j\rangle$ has the lowest weight j in the Hilbert space $\mathcal{D}^+(j)$, while the state $|j,-j\rangle$ has the highest weight $-j$ in the Hilbert space $\mathcal{D}^-(j)$. Also, there exist principal continuos representations on the Hilbert space

$$\mathcal{C}_\alpha(\varsigma)=\left\{|\varsigma,\alpha;m_\alpha\rangle:\quad \varsigma\in I\!R^+;\quad 0\le\alpha<1;\quad m_\alpha=\alpha+n,\quad n\in Z\right\} \tag{84}$$

where

$$\mathbf{J}^2 \left|\varsigma, \alpha; m_\alpha\right\rangle = \left(\varsigma^2 + \frac{1}{4}\right) \left|\varsigma, \alpha; m_\alpha\right\rangle , \qquad J_3 \left|\varsigma, \alpha; m_\alpha\right\rangle = m_\alpha \left|\varsigma, \alpha; m_\alpha\right\rangle . \tag{85}$$

Furtheremore, the Lie algebra $su(1,1)$ has complementary continuos representations, which are realized in the Hilbert space as

$$\begin{aligned} \mathcal{C}_\alpha(\tau) = \{ \left|\tau, \alpha; m_\alpha\right\rangle : \quad & -\frac{1}{2} < \tau < -\alpha, \quad 0 \leq \alpha < \frac{1}{2} \\ & \text{or } -\frac{1}{2} < \tau < \alpha - 1, \quad \frac{1}{2} < \alpha < 1; \quad m_\alpha = \alpha + n, \quad n \in Z \} \end{aligned} \tag{86}$$

where

$$\mathbf{J}^2 \left|\tau, \alpha; m_\alpha\right\rangle = -\tau\left(\tau + 1\right) \left|\tau, \alpha; m_\alpha\right\rangle , \qquad J_3 \left|\tau, \alpha; m_\alpha\right\rangle = m_\alpha \left|\tau, \alpha; m_\alpha\right\rangle . \tag{87}$$

It is clear that the eigenvalues of the Casimir operator $\mathbf{J}^2$ corresponding to the principal and complementary continuos representations are the same if we choose $\tau = -\frac{1}{2} + i\varsigma$. It is well known that the decomposition of the tensor product of positive (or negative) discrete representations of the Lie algebra $su(1,1)$ is

$$\mathcal{D}^\pm(j) \otimes \mathcal{D}^\pm(s) \cong \bigoplus_{l \geq j+s} \mathcal{D}^\pm(l) . \tag{88}$$

And, for $j \geq s$ the decomposition of the discrete representations become

$$\begin{aligned} \mathcal{D}^+(s) \otimes \mathcal{D}^-(j) &\cong \int_0^{\oplus\infty} \mathcal{C}_{s-j+\mu}(\varsigma) d\varsigma & & j - s \leq \frac{1}{2} \quad \text{and} \quad j + s \geq \frac{1}{2} \\ \mathcal{D}^+(s) \otimes \mathcal{D}^-(j) &\cong \int_0^{\oplus\infty} \mathcal{C}_{s-j+\mu}(\varsigma) d\varsigma \oplus \mathcal{C}_{s-j+\mu}(\tau = -j - s) & & j + s < \frac{1}{2} \\ \mathcal{D}^+(s) \otimes \mathcal{D}^-(j) &\cong \int_0^{\oplus\infty} \mathcal{C}_{s-j+\mu}(\varsigma) d\varsigma \oplus \bigoplus_{\frac{1}{2} < l \leq j - s} \mathcal{D}^-(l) & & j - s > \frac{1}{2} , \end{aligned} \tag{89}$$

where $j - s \leq \mu < j - s + 1$, and all the direct sums are in integer steps. Suppose the kets $|j, m_j\rangle$ span the two principal discrete representation spaces of the operator $\mathbf{J}$, as given in (57) as (83). According to the decomposition (88) and (89), and since $m_l = m_j \pm \frac{1}{2}$, the allowed discrete representations of $\mathbf{L}^2$ with the spectrum $\mathbf{L}^2 = -(j + m - \frac{1}{2})(j + m - \frac{3}{2})$ is in $\mathcal{D}^\pm(l = j + m - \frac{1}{2})$. Note that $m \geq 0$ for both possitive and negative principal discrete representations in (88), and $1 - j < m \leq -1$ for the negative principal discrete representation in the last equation of (89). If we denote the eigenvalues of D by $\lambda_{j,m}$, using Eqs. (76), the spectrum of the squared Dirac operator is then calculated as

$$\lambda_{j,m}^2 = \left(j + m - \frac{1}{2}\right)^2 \left[1 + \frac{1 - \left(j + m - \frac{1}{2}\right)^2}{N(N-2)}\right] + m(2j + m - 1)\left[\frac{4\left(j + m - 1\right)^2 - 1}{2N(N-2)} - 1\right] , \tag{90}$$

where the operators $\mathbf{J}^2$ and $\mathbf{L}^2$ are restricted over their common eigenstates. According to the above equation the spectrum of the Dirac operator on noncommutative AdS_2 depends on both quantum numbers j and m, whereas in the commutative case (hyperbolic plane) the spectrum depends only on j. It is interesting to see whether the commutative spectrum can be reobtained from the above noncommutative expression. In the limit $N \to \infty$, observe that $\lambda_{j,m}^2 \to \left(j - \frac{1}{2}\right)^2$ which depends only on the quantum number j, and this is consistent with the fact that the spectrum of the Dirac operator on AdS_2 is degenerate. Hence, noncommutativity has lifted the degeneracy. Finally note that for a finite N, the state $j = N - \frac{1}{2}$ and $m = 0$ is a zeromode, and if we require the spectrum $\lambda_{j,m}$ to be real, we find that $j + m \leq N - \frac{1}{2}$.

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